Random Planar Lattices and Integrated SuperBrownian Excursion

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ABSTRACT: *In this extended abstract, a surprising connection is described between a specific brand of random lattices, namely planar quadrangulations, and Aldous' Integrated SuperBrownian Excursion (ISE). As a consequence, the radius r_n of a random quadrangulation with n faces is shown to converge, up to scaling, to the width $r = R - L$ of the support of the one-dimensional ISE, or more precisely:*

$$n^{-1/4} r_n \xrightarrow{\text{law}} (8/9)^{1/4}\, r.$$

The combinatorial ingredients are an encoding by well labelled trees, *reminiscent of the work of Cori and Vauquelin, and the* conjugation of tree principle, *used to relate the latter trees to embedded (discrete) plane trees in the sense of Aldous.*

From probability, we need a new result of independent interest, namely the weak convergence of the encoding of a random embedded plane tree by two contour walks $(e^{(n)}, \hat{W}^{(n)})$ to the Brownian snake description $(e, \hat{W})$ of ISE.

1 Introduction

From a distant perspective, this article uncovers a surprising, and hopefully deep, relation between two famous models: *random planar maps*, as studied in combinatorics and mathematical quantum physics, and *Brownian snakes*, as studied in probability and mathematical statistical physics. More precisely, our results relate some distance-related functionals of *random quadrangulations* to functionals of Aldous' *Integrated SuperBrownian Excursion* (ISE) in dimension one.

In this extended abstract, proofs are omitted due to space limitations.

Quadrangulations. On the one hand, quadrangulations are finite plane graphs with four-regular faces (see Section 2 for precise definitions). Random quadrangulations, like random triangulations, random polyhedra, or physicists' ϕ^4-models, are instances of a general family of random lattices that has received considerable attention both in combinatorics (under the name *random planar maps*, following Tutte's terminology [26]) and in physics (under the name *Euclidean two-dimensional discretised quantum geometry*, or more simply *dynamical triangulations* or *fluid lattices* [2, 8, 14]).

Many probabilistic properties of random planar maps have been studied, that are *local properties* like vertex or face degrees [6, 13], or $0-1$ laws for properties expressible in first order logic [7]. Other well documented families of properties are related to connectedness and constant size separators [5], also known as branchings into baby universes [17]. In this article we consider another fundamental aspect of the geometry of random maps, namely *global properties of distances.* The *profile* $(H_k^n)_{k\geq 0}$ and *radius* r_n of a random quadrangulation with n faces are defined in analogy with the classical profile and height of trees: H_k^n is the number of vertices at distance k of a basepoint, while r_n is the maximal distance reached. The profile was studied (with triangulations instead of quadrangulations) by physicists

Watabiki, Ambjørn et *al.* [27, 3] who gave a consistency argument proving that the only possible scaling for the profile is $k \sim n^{1/4}$, a property which reads in their terminology *the internal Hausdorff dimension is 4*. Independently the conjecture that $\mathbb{E}(r_n) \sim cn^{1/4}$ was proposed by Schaeffer [24].

Integrated SuperBrownian Excursion. On the other hand, ISE was introduced by Aldous as a model of random distributions of masses [1]. He considers random embedded discrete trees as obtained by the following two steps: first an abstract tree t, say a Cayley tree with n nodes, is taken from the uniform distribution and each edge of t is given length 1; then t is embedded in the regular lattice on $\mathbb{Z}^d$, with the root at the origin, and edges of the tree randomly mapped on edges of the lattice. Assigning masses to leaves of the tree t yield a random distribution of mass on $\mathbb{Z}^d$. Upon scaling the lattice to $n^{-1/4}\mathbb{Z}^d$, these random distributions of mass admit, for n going to infinity, a continuum limit $\mathcal{J}$ which is a random probability measure on $\mathbb{R}^d$ called ISE.

Derbez and Slade proved that ISE describes in dimension larger than eight the continuum limit of a model of lattice trees [12], while Hara and Slade obtained the same limit for the incipient infinite cluster in percolation in dimension larger than six [15]. As opposed to these works, we shall consider ISE in dimension one and our embedded discrete trees should be thought of as awfully folded on a line. The support of ISE is then a random interval (L, R) of $\mathbb{R}$ that contains the origin.

From quadrangulation to ISE. The purpose of this paper is to draw a relation between, on the one hand, random quadrangulations, and, on the other hand, Aldous' ISE. This relation implies in particular that the radius r_n of random quadrangulations, upon proper scaling, weakly converges to the width of the support of ISE in one dimension, that is the continuous random variable $r = R - L$. We shall indeed prove (Corollary 5.2) that

$$n^{-1/4} r_n \xrightarrow{\text{law}} (8/9)^{1/4}\, r.$$

While this proves the conjecture $\mathbb{E}(r_n) \sim cn^{1/4}$, the value of the constant c remains unknown because, as mentioned by Aldous [1], nothing is known on R or $R - L$.

The path from quadrangulations to ISE consists of three main steps, the first two of combinatorial nature and the last with a more probabilistic flavor. Our first result, Theorem 3.1, revisits a correspondence of Cori and Vauquelin [10] between planar maps and some *well labelled trees*, that can be viewed as plane trees embedded in the positive half-line. Thanks to an alternative construction from [24, Chap. 7], we show that in this correspondence the profile can be mapped to the mass distribution on the half-line.

Safe the positivity condition, well labelled trees would be constructed exactly according to Aldous' prescription for embedded discrete trees. Our second step, Theorem 4.2, thus consists in removing the conditioning. More precisely, using the *conjugation of tree principle* of [24, Chap. 2], we bound the discrepancy between the mass distribution of our conditioned trees on the half-line and a translated mass distribution of freely embedded trees. In particular the radius r_n is attached to the width of the support of random freely embedded trees:

$$|r_n - (R_n - L_n)| \leq 2.$$

Since our freely embedded trees are constructed according to Aldous' prescription, one could expect to be able to conclude directly.

Contour walks and Brownian snakes. A first objection is that the construction of ISE as a continuum limit of mass distributions supported by embedded discrete trees was only outlined in Aldous' original paper. The original mathematical definition is by embedding a continuum random tree (CRT), which amounts to exchanging the embedding and the continuum limit. But Borgs et *al.* proved that indeed ISE is the limit of mass distributions supported by embedded Cayley trees [9] and their proof could certainly be adapted to other simple classes of trees and in particular to our embedded plane trees.

However a second, more important, objection is that weak convergence of probability measures is not adequate to our purpose, since we are interested in particular in convergence of the width of the support, *which is not a continuous functional on the space of measures.* In order to override this objection, we turn to the description of ISE in terms of superprocesses: ISE can be constructed from the Brownian snake with lifetime e, the standard Brownian excursion [1, 19].

From the discrete point of view, we consider the encoding of an embedded plane tree by a couple of contour walks (x_k, y_k), that encode respectively the height of the node visited at time k and its position on the line. Our last result, Theorem 5.1, is the weak convergence, upon proper scaling, of this couple of walks to the Brownian snake with lifetime e:

$$\big(e^{(n)}(s), \hat{W}^{(n)}(s)\big) \xrightarrow{\text{law}} \big(e(s), \hat{W}_s\big).$$

Finally as $R = \max_s \hat{W}_s$ and $L = \min_s \hat{W}_s$ this convergence result is sufficient to conclude on the radius and to deal with other functionals like the profile.

2 Combinatorial and probabilistic models

2.1 Planar maps and quadrangulations

A *planar map* is a proper embedding (without edge crossings) of a connected graph in the plane. Loops and multiple edges are *a priori* allowed. A planar map is *rooted* if there is a *root*, *i.e.* a distinguished edge on the border of the infinite face, which is oriented counterclockwise. The origin of the root is called the *root vertex.* Two rooted planar maps are considered identical if there exists an homeomorphism *of the plane* that sends one map onto the other (roots included).

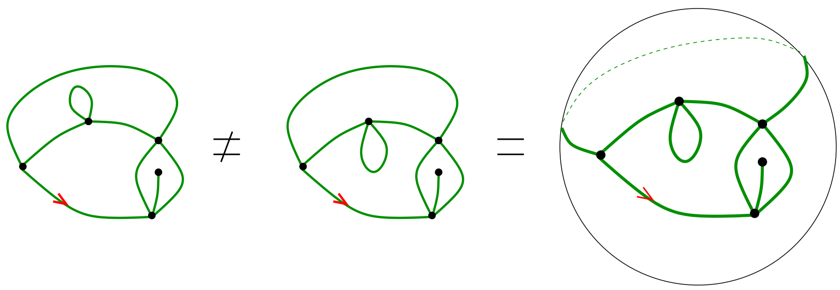

Figure 1: Two distinct planar maps, and a spherical representation of the second.

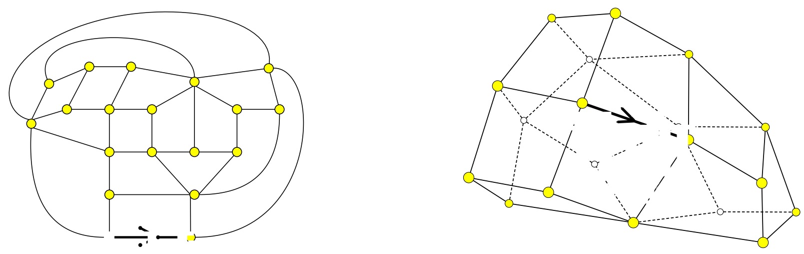

Figure 2: Random quadrangulations, in planar or spherical representation.

The difference between planar graphs and planar maps is that the cyclic order of edges around vertices matters in maps, as illustrated by Figure 1. Observe that planar maps can be equivalently defined on the sphere. In particular Euler's characteristic formula applies and provides a relation between the numbers n of edges, f of faces and v of vertices of any planar map: $f + v = n + 2$.

The *degree* of a face or of a vertex of a map is its number of incidence of edges. A planar map is a *quadrangulation* if all faces have degree four. All (planar) quadrangulations are *bipartite*: their vertices can be colored in black or white so that the root is white and any edge joins two vertices with different colors. In particular a quadrangulation contains no loop but may contain multiple edges. See Figures 2 and 3 for examples of quadrangulations.

2.2 Random planar lattices

Let $\mathcal{Q}_n$ denote the set of rooted quadrangulations with n faces. A quadrangulation with n faces has $2n$ edges (because of the degree constraint) and $n+2$ vertices (applying Euler's formula).

Let L_n be a random variable with uniform distribution on $\mathcal{Q}_n$. More formally, L_n is the $\mathcal{Q}_n$-valued random variable such that for all $Q \in \mathcal{Q}_n$

$$\Pr(L_n = Q) = \frac{1}{|\mathcal{Q}_n|}.$$

The random variable L_n is our *random planar lattice*. To explain this terminology, taken from physicists, observe that locally the usual planar square lattice is a planar map whose faces and vertices all have degree four. Our random planar lattice corresponds to a relaxation of the constraint on vertices.

Classical variants of this definition are obtained by replacing quadrangulations with n faces by triangulations with $2n$ triangles, or by (vertex-)4-regular maps with n vertices, or by all planar maps with n edges, *etc.* All these random planar lattices have been considered both in combinatorics (see [5] and references therein) and in mathematical physics (definitions are there phrased using a "symmetry weight" instead of rooted objects, see [2] and references therein). They are believed to behave exactly the same with respect to long range phenomena [23].

In this article we concentrate on quadrangulations because of their combinatorial relation to well labelled trees, as explained below.

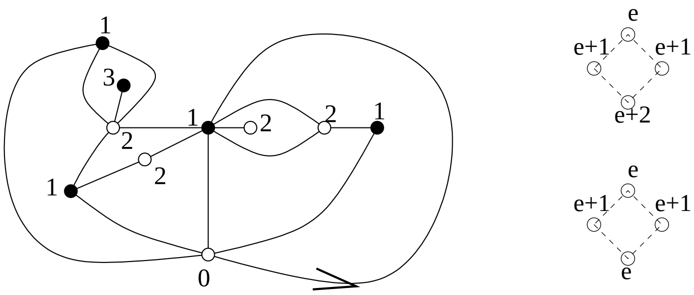

Figure 3: Labelling by distance from the root vertex and the two possible configurations of labels (top: a simple face; bottom: a confluent face).

2.3 The profile of a map

The distance $d(x, y)$ between two vertices x and y of a map is the minimal number of edges on a path from x to y (in other terms all edges have abstract length 1).

The *profile* of a rooted map M is the sequence $(H_k)_{k\geq 1}$, where $H_k \equiv H_k(M)$ is the number of vertices at distance k of the root vertex v_0. By construction the support of the profile of a rooted map is an interval *i.e.* $\{k \mid H_k > 0\} = [1, r]$ where r is the *radius* of the map (sometimes also called *eccentricity*). The radius r is closely related to the *diameter*, that is the largest distance between two vertices of a map: in particular $r \leq d \leq 2r$. The quadrangulation of Figure 3 has radius 3.

The *profile of the random planar lattice* L_n is the random variable $(H_k^{(n)})_{k\geq 1}$ that is defined by taking the profile of an instance of L_n. Similarly the radius of a random planar lattice is a positive integer valued random variable r_n.

3 Encoding the profile with well labelled trees

3.1 Well labelled trees and the encoding result

A *plane tree* is a rooted planar map without cycle (and thus with only one face). Equivalently plane trees can be recursively defined as follows:

- the smallest tree is made of a single vertex,
- any other tree is a non-empty sequence of subtrees attached to a root.

Observe that plane trees with n edges under the uniform distribution are exactly Galton-Watson trees with geometrically distributed offspring conditioned to have $n + 1$ nodes.

A plane tree is *well labelled* if all its vertices have positive integral labels, the labels of two adjacent vertices differ at most by one, and the label of the root vertex is one.

The *label distribution* of a well labelled tree is the sequence $(\lambda_k)_{k\geq 1}$ where λ_k is the number of vertices with label k in the tree. By construction the support of the label distribution is an interval: there exists an integer μ such that $\{k \mid \lambda_k >$

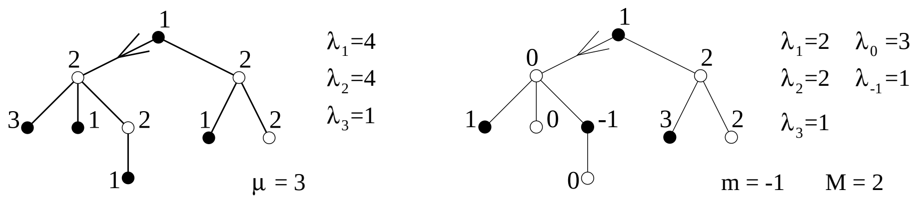

Figure 4: A well labelled tree and an unconstrained well labelled tree with their label distributions.

$0\} = [1, \mu]$. This integer μ is the maximal label of the tree. These definitions are illustrated by Figure 4 (left-hand side).

Our main encoding tool is the following theorem.

Theorem 3.1 (Cori-Vauquelin [10], Schaeffer [24]) *There exists a bijection between rooted quadrangulations with n faces and well labelled trees with n edges such that the profile $(H_k)_{k\geq 1}$ is mapped onto the label distribution $(\lambda_k)_{k\geq 1}$.*

The proof goes in three steps. First some properties of distances in quadrangulations are indicated. This allows in a second step to define the encoding, as a mapping $\mathcal{T}$ from quadrangulations to well labelled trees. A decoding procedure allows then to prove that $\mathcal{T}$ is faithful.

Let us first make a few historical comments. Well labelled trees were introduced by Cori and Vauquelin [10] to give an encoding of all planar maps with n edges. Because of a classical bijection between the latter maps and quadrangulations with n faces, their result imply the first part of Theorem 3.1. Cori and Vauquelin's bijection has been extended to bipartite maps by Arquès [4] and to higher genus maps by Marcus and Vauquelin [21]. All these constructions were recursive and based on encodings of maps with permutations (also known as rotation systems). They do not provide a clear interpretation of the λ_k.

The bijection we use here is much simpler and immediately leads to the second part of Theorem 3.1. This latter approach was extended to non separable maps by Jacquard [16] and to higher genus by Marcus and Schaeffer [20].

3.2 Properties of distances in a quadrangulation

Let Q be a rooted quadrangulation and denote v_0 its root vertex. The labelling ϕ of the map Q is defined by $\phi(x) = d(x, v_0)$ for each vertex x, where $d(x, y)$ denote the distance in Q (cf. Figure 3). This labelling satisfies the following immediate properties:

Proposition 3.2 *If x and y are joined by an edge, $|\phi(x) - \phi(y)| = 1$. Indeed the quadrangulation being bipartite, a vertex x is white if and only if $\phi(x)$ is even, black if and only if $\phi(x)$ is odd.*

Corollary 3.3 *Around a face, four vertices appear: a black x_1, a white y_1, a black x_2 and a white y_2. These vertices satisfy at least one of the two equalities $\phi(x_1) = \phi(x_2)$ or $\phi(y_1) = \phi(y_2)$ (cf. Figure 3).*

A face will be said *simple* when only one equality is satisfied and *confluent* otherwise (see Figure 3). It should be noted that one may have $x_1 = x_2$ or $y_1 = y_2$.

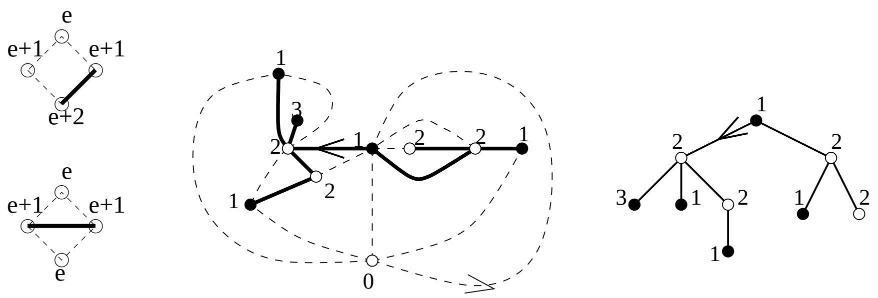

Figure 5: The rules of selection of edges and an example.

3.3 Construction of the encoding

Let Q be a rooted quadrangulation with its distance labelling. The map Q' is obtained by dividing all confluent faces Q into two triangular faces by an edge joining the two vertices with maximal label. Let us now define a subset $\mathcal{T}(Q)$ of edges of Q' by two selection rules:

- In each confluent face of Q, the edge that was added to form Q' is selected.
- For each simple face f of Q, an edge e is selected: let v be the vertex with maximal label in f, then e is the edge leaving v with f on its left.

These two selection rules are illustrated by Figure 5. The first selected edge around the endpoint of the root of Q is taken to be the root of $\mathcal{T}(Q)$.

The proof of Theorem 3.1 is now completed in two steps. First $\mathcal{T}(Q)$, which is *a priori* only defined as a subset of edges of Q' together with their incident vertices, is shown to be in fact a well labelled tree with n edges. Second the inverse mapping is described and used to prove that the mapping $\mathcal{T}$ is faithful. This proves the following proposition, which clearly implies Theorem 3.1.

Proposition 3.4 (Schaeffer [24]) *The mapping $\mathcal{T}$ is one-to-one between quadrangulations with n faces and well labelled trees with n edges.*

4 Random embedded trees

4.1 Unconstrained well labelled trees as embedded trees

A plane tree is an *unconstrained well labelled tree* if all its vertices have integral labels, the labels of two adjacent vertices differ at most by one, and the label of the root vertex is one.

The advantage of unconstrained well labelled trees over well labelled trees is that their labels are not anymore required to be positive. In particular the number of unconstrained labellings of a plane tree with n edges is just 3^n.

The label distribution of an unconstrained well labelled tree is now a sequence $(\lambda_k)_{m<k<M}$, that is supported by an interval $[m, M]$ with $m \geq 1 \geq M$. These definitions illustrated by Figure 4 (right-hand side).

Similar labellings have been considered by D. Aldous [1] with the following interpretation: the tree is folded on the line with its root at position 1 and with each edge receiving a random length (here $+1$, 0, or -1). The labels thus describe the position of nodes on the line and, upon counting the number of nodes at position j, a random mass distribution is obtained.

In view of this interpretation and for concision's sake, let us rename *unconstrained well labelled trees* and call them instead *embedded trees.*

4.2 Random trees

Let $\mathcal{W}_n$ denote the set of well labelled trees with n edges and $\mathcal{U}_n$ the set of embedded trees (unconstrained well labelled trees) with n edges. Of course $\mathcal{W}_n \subset \mathcal{U}_n$ and

$$|\mathcal{U}_n| = \frac{3^n}{n+1}\binom{2n}{n}$$

since the number of plane trees with n edges is the nth Catalan number.

Let us consider the random variables W_n and U_n with uniform distribution on these two sets (W_n is $\mathcal{W}_n$-valued and U_n is $\mathcal{U}_n$-valued). The label distribution of the corresponding random trees are two random variables that we shall denote $(\lambda_k^{(n)})_{k\geq 1}$ for random well labelled trees, and $(\Lambda_k^{(n)})_{k\in\mathbb{Z}}$ for random embedded trees. For random well labelled trees we also use the notation μ_n, and for random embedded trees the notations m_n and M_n.

On the one hand, random well labelled trees "are" random quadrangulations. For instance, the next corollary is an immediate consequence of Theorem 3.1.

Corollary 4.1 *The label distribution of random well labelled trees is equidistributed to the profile of quadrangulations:*

$$(\lambda_k^{(n)})_{k\geq 1} \overset{law}{=} (H_k^{(n)})_{k\geq 1}.$$

In particular $r_n = \mu_n$.

On the other hand, random embedded trees are defined in accordance with Aldous' prescription for ISE. Observe moreover that random well labelled trees are random embedded trees conditioned to positivity of the support. Instead of studying directly random well labelled trees we shall thus use a second combinatorial ingredient to reduce the problem to embedded trees.

4.3 How to relieve the positivity conditioning

Let us consider the following *cumulated* profile and label distributions: for $k \geq 0$,

$$\hat{H}_k = \sum_{\ell=1}^{k} H_\ell, \quad \hat{\lambda}_k = \sum_{\ell=1}^{k} \lambda_\ell, \quad \hat{\Lambda}_k = \sum_{\ell=1}^{k} \Lambda_{\ell-m-1}$$

The random variables $H_k^{(n)}$, $\hat{\lambda}_k^{(n)}$ and $\hat{\Lambda}_k^{(n)}$ are defined accordingly. (In the definition of $\hat{\Lambda}_k^{(n)}$, beware that the occurrences of m_n and $\Lambda_k^{(n)}$ are to be evaluated on a same instance of L_n.)

Theorem 4.2 *There is a coupling* (W_n, U_n) *(*i.e. *a distribution on* $\mathcal{W}_n \times \mathcal{U}_n$ *such that the marginals are* W_n *and* U_n *as previously defined) such that the induced joined distribution* $(\lambda^{(n)}, \Lambda^{(n)})$ *satisfies for all* k

$$-\Lambda_{k-1}^{(n)} - \Lambda_k^{(n)} \leq \hat{\lambda}_k^{(n)} - \hat{\Lambda}_k^{(n)} \leq \Lambda_{k+1}^{(n)} + \Lambda_{k+2}^{(n)}$$

and in particular

$$|\mu_n - (M_n - m_n)| \leq 2$$

As proven at the end of this section, Theorem 4.2 is merely a probabilistic restatement of the following combinatorial result.

Theorem 4.3 *There exists a partition of* $\mathcal{U}_n = \bigcup_{C \in \mathcal{C}_n} C$ *into disjoint* conjugacy classes *each of size at most* $n+2$ *and such that in each class* $C \in \mathcal{C}_n$

- *well labelled trees are fairly represented:*

$$2 \cdot |C| \ = \ (n+2) \cdot |C \cap \mathcal{W}_n|,$$

- *and for any* $W \in \mathcal{W}_n \cap C$, $U \in C$ *and* $k \geq 1$,

$$\hat{\Lambda}_{k-2}(U) \leq \hat{\lambda}_k(W) \leq \hat{\Lambda}_{k+2}(U).$$

Corollary 4.4 (Cori-Vauquelin, 1981) *The number of well labelled trees with* n *edges, (which is also the number of quadrangulations with* n *faces), is*

$$|\mathcal{W}_n| = \frac{2}{n+2} \cdot |\mathcal{U}_n| = \frac{2}{n+2} \cdot \frac{3^n}{n+1} \binom{2n}{n}.$$

The proof of Theorem 4.3 relies on an encoding of plane trees in terms of another family of trees, called *blossom trees*, and on the *conjugation of tree* principle. These tools were introduced in [24] for the different purpose of giving a direct combinatorial proof of Corollary 4.4.

Proof of Theorem 4.2. The distribution on $\mathcal{W}_n \times \mathcal{U}_n$ is immediately obtained from the partition $\mathcal{U}_n = \bigcup_{C \in \mathcal{C}_n} C$ as follows: for any (W, U) in $\mathcal{W}_n \times \mathcal{U}_n$, let

$$\Pr((W_n, U_n) = (W, U)) = \begin{cases} \frac{1}{2|\mathcal{U}_n|} & \text{if } U,\ W \text{ are both in } C \text{ with } |C \cap \mathcal{W}_n| = 2, \\ \frac{1}{|\mathcal{U}_n|} & \text{if } U,\ W \text{ are both in } C \text{ with } |C \cap \mathcal{W}_n| = 1, \\ 0 & \text{if } U \in C_1 \text{ and } W \in C_2 \text{ with } C_1 \neq C_2. \end{cases}$$

In view of the first part of Theorem 4.3, the marginals are uniformly distributed. The second part of Theorem 4.3 gives the two inequalities. □

5 Embedded trees and Brownian snakes

To each embedded tree U with size n is associated a bidimensional path, with length $2n$, starting from and arriving at $(0,0)$, and whose value at step k is (x_k, y_k), in which x_k is the height (distance from the root) of the vertex v_k visited at step k in the contour walk around the tree, and y_k is the label of the same vertex v_k.

The path $(x_k)_{0\le k\le 2n}$ is the *Dyck path* associated to the tree U, also called *contour process* in [19, Ch. I.3]. For each well labelled tree U with size n, set

$$\Xi_n(U) \quad = \quad \left(\frac{x_{\lfloor 2ns\rfloor}(U)}{\sqrt{2n}}, \frac{y_{\lfloor 2ns\rfloor}(U)\,\sqrt{3}}{(8n)^{1/4}}\right)_{0\le s\le 1}$$

Turning now to the uniform distribution on embedded trees with size n, we define three random variables $X^{(n)}$, $e^{(n)}$, $\hat{W}^{(n)}$, with values in the Skorohod spaces $D([0,1],\mathbb{R}^2)$ (resp. $D([0,1],\mathbb{R})$:

$$X^{(n)} \quad \equiv \quad \left(e^{(n)}, \hat{W}^{(n)}\right) \quad = \quad \Xi_n(U_n).$$

As was proved by Kaigh [18], the scaled version of the contour process, $e^{(n)}$, converges weakly to the normalised Brownian excursion e. In this section we extend Kaigh's result to $X^{(n)}$. Let

$$W = (W_s(t))_{0\le s\le 1,\ 0\le t\le e(s)}$$

be the Brownian snake with lifetime e, as studied previously in [1, 9, 11, 12, 19, 25]. That is, $t \to W_s(t)$ is a standard Brownian motion defined for $0 \le t \le e(s)$, and $s \to W_s(.)$ is a path-valued Markov process with transition function described as follows: for $s_1 < s_2$, and for $m = \inf_{s_1\le u\le s_2}\ e(u)$, conditionally given $W_{s_1}(.)$, on the one hand we have that

$$(W_{s_1}(t))_{0\le t\le m} = (W_{s_2}(t))_{0\le t\le m}\,,$$

and on the other hand $(W_{s_2}(m+t))_{0\le t\le e(t_2)-m}$ is a standard Brownian motion starting from $W_{s_2}(m)$, independent of $W_{s_1}(.)$. Set

$$\hat{W}_s = W_s(e(s)), \quad X_s = \left(e(s), \hat{W}_s\right).$$

Theorem 5.1 *$X^{(n)}$ converges weakly to $X = (X_s)_{0\le s\le 1}$.*

According to Corollary 4.1 and to Theorem 4.2, the radius r_n of the quadrangulation corresponding to U_n satisfies

$$\left|(8/9)^{1/4}\left(\max_{0\le s\le 1} \hat{W}_s^{(n)} - \min_{0\le s\le 1} \hat{W}_s^{(n)}\right) - n^{-1/4}\,r_n\right| \le 2n^{-1/4}.$$

Theorem 5.1 thus not only proves the conjecture $\mathbb{E}(r_n) = \Theta(n^{1/4})$ but leads to a much more precise characterization:

Corollary 5.2 *The random variable $n^{-1/4}\,r_n$ converges weakly to $(8/9)^{1/4}\,r$, in which*

$$r = \max_{0\le t\le 1} \hat{W}_t - \min_{0\le t\le 1} \hat{W}_t.$$

Let $\mathcal{J}_n$ denote the empirical measure of labels of a random embedded tree:

$$\mathcal{J}_n = \frac{1}{n}\sum_k \Lambda_k \delta_k.$$

Following Aldous [1], for simple families of trees $\mathcal{J}_n$ is expected to converge upon scaling to a random mass distribution $\mathcal{J}$ called ISE, related to W through

$$\int g \; d\mathcal{J} = \int_0^1 g\left(\hat{W}_s\right) \, ds, \tag{1}$$

see [19, Ch. IV.6]. In [9] the convergence of $\mathcal{J}_n$ to $\mathcal{J}$ is proved for random embedded Cayley trees. Although these trees are not exactly our random embedded *plane* trees, the proof could certainly be adapted.

In view of Relation (1) the limiting radius r is the width of the support of $\mathcal{J}$, that is

$$r = R - L$$

in the notations of [1, Section 3.2]. However, r is not a continuous functional of $\mathcal{J}$, so that the weak convergence of $\mathcal{J}_n$ to $\mathcal{J}$, as obtained in [9], would not be sufficient to yield Corollary 5.2.

References

[1] D. J. Aldous. Tree-based models for random distribution of mass. *J. Statist. Phys.*, 73(3-4):625–641, 1993.

[2] J. Ambjørn, B. Durhuus and T. Jónsson. *Quantum gravity, a statistical field theory approach.* Cambridge Monographs on Mathematical Physics, 1997.

[3] J. Ambjørn and Y. Watabiki, Scaling in Quantum Gravity. *Nucl. Phys. B*, 445:129–144, 1995.

[4] D. Arquès. Les hypercartes planaires sont des arbres très bien étiquetés. *Discrete Math.*, 58:11–24, 1986.

[5] C. Banderier, P. Flajolet, G. Schaeffer and M. Soria. Random Maps, Coalescing Saddles, Singularity Analysis, and Airy Phenomena. *Random Struct. & Algorithms,* 19:194-246, 2001.

[6] E. A. Bender and E. R. Canfield. Face Sizes of 3-polytopes. *J. Combinat. Theory, Ser. B*, 46:58–65, 1989.

[7] E. A. Bender, K. J. Compton and L. B. Richmond. 0–1 laws for maps. *Random Struct. & Algorithms*, 14:215–237, 1999.

[8] E. Brezin, C. Itzykson, G. Parisi and J.-B. Zuber. Planar Diagrams. *Comm. Math. Phys.*, 59:35–47, 1978.

[9] C. Borgs, J. Chayes, R. van der Hofstad and G. Slade. Mean-field lattice trees. On combinatorics and statistical mechanics. *Ann. Comb.*, 3(2-4):205–221, 1999.

[10] R. Cori and B. Vauquelin. Planar maps are well labeled trees. *Canad. J. Math.*, 33(5):1023–1042, 1981.

[11] A. Dembo and O. Zeitouni. Large deviations for random distribution of mass. *Random discrete structures (Minneapolis, MN, 1993)*, 45–53, IMA Vol. Math. Appl., 76, Springer, New York, 1996.

[12] E. Derbez and G. Slade. The scaling limit of lattice trees in high dimensions. *Comm. Math. Phys.*, 193(1):69–104, 1998.

[13] Z. Gao and N. C. Wormald. The Distribution of the Maximum Vertex Degree in Random Planar Maps. *J. Combinat. Theory, Ser. A*, 89:201–230, 2000.

[14] D. Gross, T. Piran and S. Weinberg. *Two dimensional quantum gravity and random surfaces.* World Scientific, 1992.

[15] T. Hara and G. Slade. The incipient infinite cluster in high-dimensional percolations. *Electron. Res. Announc. Amer. Math. Soc.*, 4:48-55, 1998.

[16] B. Jacquard. *Cartes et Arbres : énumération, génération et dessins.* PhD thesis, École Polytechnique, 1997, Palaiseau.

[17] S. Jain and S. D. Mathur. World sheet geometry and baby universes in 2-d quantum gravity. *Phys. Lett. B*, 305:208–213, 1993.

[18] W. D. Kaigh. An invariance principle for random walk conditioned by a late return to zero. *Ann. Prob.*, 4(1):115–121, 1976.

[19] J.-F. Le Gall, *Spatial branching processes, random snakes and partial differential equations.* Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1999.

[20] M. Marcus and G. Schaeffer. Une bijection simple pour les cartes orientables. Manuscript, 10pp, july 2001.

[21] M. Marcus and B. Vauquelin. Un codage des cartes de genre quelconque. In B. Leclerc and J.Y. Thibon, ed., *Séries formelles et combinatoire algébrique, 7ème colloque*, 399–416. Université de Marne-la-Vallée, 1995.

[22] V. V. Petrov. *Sums of Independant Random Variables.* Springer, 1975.

[23] L. B. Richmond and N. C. Wormald. Almost all maps are asymmetric. *J. of Combinat. Theory, Ser. B*, 63(1):1–7, 1995.

[24] G. Schaeffer. *Conjugaison d'arbres et cartes combinatoires aléatoires.* PhD. thesis, Université Bordeaux I, 1998, Bordeaux.

[25] L. Serlet. A large deviation principle for the Brownian snake, *Stochastic Process. Appl.*, 67(1):101–115, 1997.

[26] W.T. Tutte. A census of planar maps. *Canad. J. Math.*, 15:249–271, 1963.

[27] Y. Watabiki. Construction of noncritical string field theory by transfert matrix formalism in dynamical triangulations. *Nucl. Phys. B*, 441:119–166, 1995.

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